

# Incomplete $q$ -Chebyshev Polynomials

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## Abstract

In this paper, we get the generating functions of  $q$ -Chebyshev polynomials using  $\eta_z$  operator, which is  $\eta_z(f(z)) = f(qz)$  for any given function  $f(z)$ . Also considering explicit formulas of  $q$ -Chebyshev polynomials, we give new generalizations of  $q$ -Chebyshev polynomials called incomplete  $q$ -Chebyshev polynomials of the first and second kind. We obtain recurrence relations and several properties of these polynomials. We show that there are connections between incomplete  $q$ -Chebyshev polynomials and the some well-known polynomials.

**Keywords:**  $q$ -Chebyshev polynomials,  $q$ -Fibonacci polynomials, Incomplete polynomials, Fibonacci number

**Mathematics Subject Classification:** 11B39, 05A30

## 1 Introduction

Chebyshev polynomials are of great importance in many area of mathematics, particularly approximation theory. The Chebyshev polynomials of the second kind can be expressed by the formula

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad n \geq 2,$$

with initial conditions  $U_0 = 1$ ,  $U_1(x) = 2x$  and the Chebyshev polynomials of the first kind can be defined as

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad n \geq 2,$$

with initial conditions  $T_0(x) = 1$ ,  $T_1(x) = x$  in [10].

The well-known Fibonacci and Lucas sequences are defined by the recurrence relations

$$F_{n+1} = F_n + F_{n-1} \quad n \geq 1$$

$$L_{n+1} = L_n + L_{n-1} \quad n \geq 1$$

with initial conditions  $F_0 = 0$ ,  $F_1 = 1$  and  $L_0 = 2$ ,  $L_1 = 1$ , respectively. In [9], Filipponi introduced a generalization of the Fibonacci numbers. Accordingly, the incomplete Fibonacci and Lucas numbers are determined by:

$$F_n(k) = \sum_{j=0}^k \binom{n-1-j}{j}, \quad 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor \quad (1.1)$$

and

$$L_n(k) = \sum_{j=0}^k \frac{n}{n-j} \binom{n-j}{j}, \quad 0 \leq k \leq \lfloor \frac{n}{2} \rfloor \quad (1.2)$$

where  $n \in \mathbb{N}$ . Note that  $F_n(\lfloor \frac{n-1}{2} \rfloor) = F_n$  and  $L_n(\lfloor \frac{n}{2} \rfloor) = L_n$ . In [11], the generating functions of incomplete Fibonacci and Lucas polynomials were given by Pintér and Srivastava. For more results on the incomplete Fibonacci numbers, the readers may refer to [7, 8, 12–14].

We need  $q$ -integer and  $q$ -binomial coefficient. There are several equivalent definition and notation for the  $q$ -binomial coefficients [1, 16–18]. Let  $q \in \mathbb{C}$  with  $0 < |q| < 1$  as an indeterminate and nonnegative integer  $n$ . The  $q$ -integer of the number  $n$  is defined by

$$[n]_q := \frac{1 - q^n}{1 - q},$$

with  $[0]_q = 0$ . The  $q$ -factorial is defined by

$$[n]_q! := \begin{cases} [n]_q [n-1]_q \cdots [1]_q, & \text{if } n = 1, 2, \dots \\ 1, & \text{if } n = 0. \end{cases}$$

The Gaussian or  $q$ -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!}, \quad 0 \leq k \leq n$$

or

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_k}, \quad 0 \leq k \leq n$$

with  $\begin{bmatrix} n \\ k \end{bmatrix} = 0$  for  $n < k$ , where  $(x; q)_n$  is the  $q$ -shifted factorial, that is,  $(x; q)_0 = 1$ ,

$$(x; q)_n = \prod_{i=0}^{n-1} (1 - q^i x).$$

The  $q$ -binomial coefficient satisfies the recurrence relations and properties:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \quad (1.3)$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \quad (1.4)$$

$$\frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q \quad (1.5)$$

$$q^k \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q + q^n \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q. \quad (1.6)$$

The  $q$ -analogues of Fibonacci polynomials are studied by Carlitz in [2]. Also, a new  $q$ -analogue of the Fibonacci polynomials is defined by Cigler and obtain some of its properties in [5]. In [6], Pan study some arithmetic properties of the  $q$ -Fibonacci numbers and the  $q$ -Pell numbers. Cigler defined  $q$ -analogues of the Chebyshev polynomials and study properties of these polynomials in [3, 4].

In this paper, we derive generating functions of  $q$ -Chebyshev polynomials of first and second kind. More generally, we define incomplete  $q$ -Chebyshev polynomials of first and second kind. We get recurrence relations and several properties of these polynomials. We show that there are the relationships between  $q$ -Chebyshev polynomials and incomplete  $q$ -Chebyshev polynomials.

## 2 $q$ -Chebyshev Polynomials

**Definition 1.** The  $q$ -Chebyshev polynomials of the second kind are defined by

$$\mathcal{U}_n(x, s, q) = (1 + q^n)x \mathcal{U}_{n-1}(x, s, q) + q^{n-1}s \mathcal{U}_{n-2}(x, s, q) \quad n \geq 2, \quad (2.1)$$

with initial conditions  $\mathcal{U}_0(x, s, q) = 1$  and  $\mathcal{U}_1(x, s, q) = (1 + q)x$  in [3].

**Definition 2.** The  $q$ -Chebyshev polynomials of the first kind are defined by

$$\mathcal{T}_n(x, s, q) = (1 + q^{n-1})x \mathcal{T}_{n-1}(x, s, q) + q^{n-1}s \mathcal{T}_{n-2}(x, s, q) \quad n \geq 2, \quad (2.2)$$

with initial conditions  $\mathcal{T}_0(x, s, q) = 1$  and  $\mathcal{T}_1(x, s, q) = x$  in [3].

It is clear that  $\mathcal{U}_n(x, -1, 1) = U_n(x)$  and  $\mathcal{T}_n(x, -1, 1) = T_n(x)$ . The  $q$ -Chebyshev polynomials of the second kind is determined as the combinatorial sum

$$\mathcal{U}_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j}, \quad n \geq 0 \quad (2.3)$$

and the  $q$ -Chebyshev polynomials of the first kind is determined as

$$\mathcal{T}_n(x, s, q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j-1}}{(-q; q)_j} s^j x^{n-2j}, \quad n > 0 \quad (2.4)$$

with  $\mathcal{T}_0(x, s, q) = 1$  in [3].

Table 1: Some special cases of the  $q$ -Chebyshev polynomials of the second kind

$x$	$s$	$q$	$\mathcal{U}_n(x, s, q)$	$q$ -Chebyshev polynomials of the second kind
$x$	$-1$	$1$	$U_n(x)$	Chebyshev polynomials of the second kind
$\frac{x}{2}$	$1$	$1$	$F_{n+1}(x)$	Fibonacci polynomials
$\frac{1}{2}$	$1$	$1$	$F_{n+1}$	Fibonacci numbers
$x$	$1$	$1$	$P_{n+1}(x)$	Pell polynomials
$1$	$1$	$1$	$P_{n+1}$	Pell numbers
$\frac{1}{2}$	$2y$	$1$	$J_{n+1}(y)$	Jacobsthal polynomials
$\frac{1}{2}$	$2$	$1$	$J_{n+1}$	Jacobsthal numbers

Table 2: Some special cases of the  $q$ -Chebyshev polynomials of the first kind

$x$	$s$	$q$	$\mathcal{T}_n(x, s, q)$	$q$ -Chebyshev polynomials of the first kind
$x$	$-1$	$1$	$T_n(x)$	Chebyshev polynomials of the first kind
$\frac{x}{2}$	$1$	$1$	$\frac{1}{2} L_n(x)$	Lucas polynomials
$\frac{1}{2}$	$1$	$1$	$\frac{1}{2} L_n$	Lucas numbers
$x$	$1$	$1$	$\frac{1}{2} Q_n(x)$	Pell-Lucas polynomials
$1$	$1$	$1$	$\frac{1}{2} Q_n$	Pell-Lucas numbers
$\frac{1}{2}$	$2y$	$1$	$\frac{1}{2} j_n(y)$	Jacobsthal-Lucas polynomials
$\frac{1}{2}$	$2$	$1$	$\frac{1}{2} j_n$	Jacobsthal-Lucas numbers

## 2.1 Generating Functions of $q$ -Chebyshev Polynomials

Andrews [15] obtain the generating function for Schur's polynomials, which is defined by  $S_n(q) = S_{n-1}(q) - q^{n-2}S_{n-2}(q)$  for  $n > 1$  with initial conditions  $S_0(q) = 0$  and  $S_1(q) = 1$ . The generating funtions of  $S_n(q)$  is

$$\sum_{n=0}^{\infty} S_n(q) x^n = \frac{x}{1 - x - x^2 \eta_z} \quad (2.5)$$

where  $\eta_z$  is an operator on functions of  $z$  defined by  $\eta_z(f(z)) = f(qz)$  in [15]. We give the following theorems for generating functions of  $q$ -Chebyshev polynomials of the second and first kind with an operator  $\eta_z$ .

**Theorem 1.** *The generating function of  $q$ -Chebyshev polynomials of the second kind is*

$$G(z) = \frac{1}{1 - zx - (xqz + sqz^2) \eta_z}. \quad (2.6)$$

*Proof.* Let

$$G(z) = \sum_{n=0}^{\infty} \mathcal{U}_n z^n.$$

Now we show that

$$(1 - xz - (xqz + sqz^2) \eta_z) G(z) = 1.$$

Thus we write

$$\begin{aligned} (1 - xz - (xqz + sqz^2) \eta_z) G(z) &= \sum_{n=0}^{\infty} \mathcal{U}_n z^n - x \sum_{n=0}^{\infty} \mathcal{U}_n z^{n+1} - x \sum_{n=0}^{\infty} \mathcal{U}_n q^{n+1} z^{n+1} - s \sum_{n=0}^{\infty} \mathcal{U}_n q^{n+1} z^{n+2} \\ &= \sum_{n=0}^{\infty} \mathcal{U}_n z^n - x \sum_{n=1}^{\infty} (1 + q^n) \mathcal{U}_{n-1} z^n - s \sum_{n=2}^{\infty} \mathcal{U}_{n-2} q^{n-1} z^n \\ &= \mathcal{U}_0 + \mathcal{U}_1 z - x(1 + q) \mathcal{U}_0 z + \sum_{n=2}^{\infty} (\mathcal{U}_n - x(1 + q^n) \mathcal{U}_{n-1} - s \mathcal{U}_{n-2} q^{n-1}) z^n. \end{aligned}$$

Therefore we have from Eq. (2.1)

$$(1 - xz - (xqz + sqz^2) \eta_z) G(z) = \mathcal{U}_0 + \mathcal{U}_1 z - x(1 + q) \mathcal{U}_0 z.$$

From  $\mathcal{U}_0 = 1$  ve  $\mathcal{U}_1 = (1 + q)x$ , we get

$$(1 - xz - (xqz + sqz^2) \eta_z) G(z) = 1.$$

□

**Theorem 2.** *The generating function of  $q$ -Chebyshev polynomials of the first kind is*

$$S(z) = \frac{1 - xz}{1 - xz - (xz - sqz^2) \eta_z}. \quad (2.7)$$

*Proof.* Let  $S(z) = \sum_{n=0}^{\infty} \mathcal{T}_n z^n$ . Then

$$\begin{aligned} (1 - xz - (xz - sqz^2) \eta_z) S(z) &= \sum_{n=0}^{\infty} \mathcal{T}_n z^n - x \sum_{n=1}^{\infty} \mathcal{T}_{n-1} z^n - x \sum_{n=1}^{\infty} \mathcal{T}_{n-1} q^{n-1} z^n - s \sum_{n=2}^{\infty} \mathcal{T}_{n-2} q^{n-1} z^n \\ &= \sum_{n=0}^{\infty} \mathcal{T}_n z^n - x \sum_{n=1}^{\infty} (1 + q^{n-1}) \mathcal{T}_{n-1} z^n - s \sum_{n=2}^{\infty} \mathcal{T}_{n-2} q^{n-1} z^n \\ &= \mathcal{T}_0 + \mathcal{T}_1 z - 2x \mathcal{T}_0 z + \sum_{n=2}^{\infty} (\mathcal{T}_n - x(1 + q^{n-1}) \mathcal{T}_{n-1} - s q^{n-1} \mathcal{T}_{n-2}) z^n \end{aligned}$$

and we get using Eq. (2.2)

$$(1 - xz - (xz - sqz^2) \eta_z) S(z) = \mathcal{T}_0 + \mathcal{T}_1 z - 2\mathcal{T}_0 xz.$$

From  $\mathcal{T}_0 = 1$  ve  $\mathcal{T}_1 = x$ , we conclude that

$$S(z) - xzS(z) - xz \eta_z S(z) - sqz^2 \eta_z S(z) = 1 - xz,$$

finally we obtain

$$S(z) = \frac{1 - xz}{1 - xz - (xz - sqz^2) \eta_z}. \quad (2.8)$$

□

### 3 Incomplete $q$ -Chebyshev Polynomials

In this section, we define incomplete  $q$ -Chebyshev polynomials of the first and second kind. We give several properties for these polynomials.

**Definition 3.** For  $n$  is a nonnegative integer, the incomplete  $q$ -Chebyshev polynomials of the second kind are defined as

$$\mathcal{U}_n^k(x, s, q) = \sum_{j=0}^k q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j} \quad 0 \leq k \leq \lfloor \frac{n}{2} \rfloor. \quad (3.1)$$

When  $k = \lfloor \frac{n}{2} \rfloor$  in (3.1),  $\mathcal{U}_n^k(x, s, q) = \mathcal{U}_n(x, s, q)$ , we get the  $q$ -Chebyshev polynomials of the second kind in [3, 4]. Some special cases of the incomplete  $q$ -Chebyshev polynomials of the second kind are provided in Table 1.

**Definition 4.** For  $n$  is a nonnegative integer, the incomplete  $q$ -Chebyshev polynomials of the first kind are defined by

$$\mathcal{T}_n^k(x, s, q) = \sum_{j=0}^k q^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j-1}}{(-q; q)_j} s^j x^{n-2j} \quad 0 \leq k \leq \lfloor \frac{n}{2} \rfloor. \quad (3.2)$$

Some special cases of the incomplete  $q$ -Chebyshev polynomials of the first kind are provided in Table 2.

**Theorem 3.** The incomplete  $q$ -Chebyshev Polynomials of the second kind satisfy

$$\mathcal{U}_{n+2}^{k+1} = (1 + q^{n+2})x \mathcal{U}_{n+1}^{k+1} + q^{n+1}s \mathcal{U}_n^k \quad (3.3)$$

for  $0 \leq k \leq \frac{n-1}{2}$ .

*Proof.* From Eq. (3.1), we can write

$$\begin{aligned} (1 + q^{n+2})x \mathcal{U}_{n+1}^{k+1} + q^{n+1}s \mathcal{U}_n^k &= (1 + q^{n+2})x \sum_{j=0}^{k+1} q^{j^2} \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n+1-2j} \\ &\quad + q^{n+1}s \sum_{j=0}^k q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j} \\ &= \sum_{j=0}^{k+1} q^{j^2} \left\{ (1 + q^{n+2}) \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + q^{n+1}q^{-2j+1}(1 + q^j) \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right\} \\ &\quad \times \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n-2j+2} \\ &= \sum_{j=0}^{k+1} q^{j^2} \left\{ \left( \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + q^{n-2j+2} \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right) \right. \\ &\quad \left. + q^{n-j+2} \left( q^j \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q j - 1_q \right) \right\} \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n-2j+2}. \end{aligned}$$

Thus using Eq. (1.3) and Eq. (1.4), we get

$$\begin{aligned} (1 + q^{n+2})x \mathcal{U}_{n+1}^{k+1} + q^{n+1}s \mathcal{U}_n^k &= \sum_{j=0}^{k+1} q^{j^2} (1 + q^{n-j+2}) \begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n-2j+2} \\ &= \sum_{j=0}^{k+1} q^{j^2} \begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j+2}}{(-q; q)_j} s^j x^{n-2j+2} \\ &= \mathcal{U}_{n+2}^{k+1}. \end{aligned}$$

□

**Corollary 1.** *Incomplete  $q$ -Chebyshev Polynomials of the second kind satisfy the non-homogeneous recurrence relation*

$$\mathcal{U}_{n+2}^k = (1 + q^{n+2})x\mathcal{U}_{n+1}^k + q^{n+1}s\mathcal{U}_n^k - q^{n+1+k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{(-q; q)_{n-k}}{(-q; q)_k} s^{k+1} x^{n-2k}. \quad (3.4)$$

**Theorem 4.** *For  $0 \leq k \leq \frac{n+1}{2}$ , the following equality give a relationships between the incomplete  $q$ -Chebyshev polynomials of the first and second kind*

$$\mathcal{T}_{n+2}^k = x\mathcal{U}_{n+1}^k + q^{n+1}s\mathcal{U}_n^{k-1}. \quad (3.5)$$

*Proof.* Using Eq. (3.1), we obtain

$$\begin{aligned} \mathcal{U}_{n+1}^k + q^{n+1}s\mathcal{U}_n^{k-1} &= x \sum_{j=0}^k q^{j^2} \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n+1-2j} + q^{n+1}s \sum_{j=0}^{k-1} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j} \\ &= \sum_{j=0}^k q^{j^2} \left\{ \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + q^{n+1-2j+1}(1+q^j) \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right\} \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n-2j+2} \end{aligned}$$

From Eq. (1.4) and Eq. (1.6), we get

$$\begin{aligned} x\mathcal{U}_{n+1}^k + q^{n+1}s\mathcal{U}_n^{k-1} &= \sum_{j=0}^k q^{j^2} \frac{[n+2]_q}{[n-j+2]_q} \begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j+1}}{(-q; q)_j} s^j x^{n-2j+2} \\ &= \mathcal{T}_{n+2}^k. \end{aligned}$$

□

**Theorem 5.** *The incomplete  $q$ -Chebyshev polynomials of the first kind satisfy*

$$\mathcal{T}_{n+2}^{k+1} = (1 + q^{n+1})x\mathcal{T}_{n+1}^{k+1} + q^{n+1}s\mathcal{T}_n^k \quad (3.6)$$

for  $0 \leq k \leq \frac{n-1}{2}$ .

*Proof.* By using Eq. (3.3) and Eq. (3.5), we get

$$\begin{aligned} \mathcal{T}_{n+2}^{k+1} &= x\mathcal{U}_{n+1}^{k+1} + q^{n+1}s\mathcal{U}_n^k \\ &= (1 + q^{n+1})x^2\mathcal{U}_n^{k+1} + q^n s x \mathcal{U}_{n-1}^k + q^{n+1}s(1 + q^n)x\mathcal{U}_{n-1}^k + q^{2n}s^2\mathcal{U}_{n-2}^{k-1} \\ &= (1 + q^{n+1})x \{x\mathcal{U}_n^{k+1} + q^n s \mathcal{U}_{n-1}^k\} + q^{n+1}s \{x\mathcal{U}_{n-1}^k + q^{n-1}s\mathcal{U}_{n-2}^{k-1}\} \\ &= (1 + q^{n+1})x\mathcal{T}_{n+1}^{k+1} + q^{n+1}s\mathcal{T}_n^k. \end{aligned}$$

□

**Corollary 2.** *Incomplete  $q$ -Chebyshev polynomials of the first kind satisfy the non-homogeneous recurrence relation*

$$\mathcal{T}_{n+2}^k = (1 + q^{n+1})\mathcal{T}_{n+1}^k + q^{n+1}s\mathcal{T}_n^k - q^{n+1+k^2} \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{(-q; q)_{n-k-1}}{(-q; q)_k} s^{k+1} x^{n-2k}. \quad (3.7)$$

**Theorem 6.** *For  $0 \leq k \leq \frac{n+1}{2}$ , then*

$$\mathcal{T}_{n+2}^k = x\mathcal{U}_{n+1}^k(x, q^2s, q) + qs\mathcal{U}_n^{k-1}(x, q^2s, q) \quad (3.8)$$

holds.

*Proof.* We obtain from Eq. (3.1)

$$\begin{aligned}
x \mathcal{U}_{n+1}^k(x, q^2 s, q) + qs \mathcal{U}_n^{k-1}(x, q^2 s, q) &= x \sum_{j=0}^k q^{j^2} \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q \frac{(-q; q)_{n+1-j}}{(-q; q)_j} (q^2 s)^j x^{n+1-2j} \\
&\quad + qs \sum_{j=0}^{k-1} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} (q^2 s)^j x^{n-2j} \\
&= \sum_{j=0}^k q^{j^2} \left\{ q^{2j} \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + (1+q^j) \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right\} \frac{(-q; q)_{n+1-j}}{(-q; q)_j} s^j x^{n+2-2j} \\
&= \sum_{j=0}^k q^{j^2} \left\{ q^j \left\{ q^j \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right\} + \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right\} \\
&\quad \times \frac{(-q; q)_{n+1-j}}{(-q; q)_j} s^j x^{n+2-2j}.
\end{aligned}$$

By using Eq. (1.3), we have

$$\begin{aligned}
x \mathcal{U}_{n+1}^k(x, q^2 s, q) + qs \mathcal{U}_n^{k-1}(x, q^2 s, q) &= \sum_{j=0}^k q^{j^2} \left\{ q^j \begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q + \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right\} \frac{(-q; q)_{n+1-j}}{(-q; q)_j} s^j x^{n+2-2j} \\
&= \sum_{j=0}^k q^{j^2} \frac{[n+2]_q}{[n+2-j]_q} \begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q \frac{(-q; q)_{n+1-j}}{(-q; q)_j} s^j x^{n+2-2j} \\
&= \mathcal{T}_{n+2}^k.
\end{aligned}$$

□

**Theorem 7.** We have

$$(1 + q^{n+2}) \mathcal{T}_{n+2}^k = \mathcal{U}_{n+2}^k + q^{2n+3} s \mathcal{U}_n^{k-1}, \quad 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad (3.9)$$

*Proof.* From Eq. (3.3) and Eq. (3.1), we get

$$\begin{aligned}
\mathcal{U}_{n+2}^k + q^{2n+3} s \mathcal{U}_n^{k-1} &= \{(1 + q^{n+2})x \mathcal{U}_{n+1}^k + q^{n+1} s \mathcal{U}_n^{k-1}\} + q^{2n+3} s \mathcal{U}_n^{k-1} \\
&= \sum_{j=0}^k q^{j^2} \left\{ \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + q^{n+1-2j+1} (1+q^j) \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right\} \frac{(-q; q)_{n+1-j}}{(-q; q)_j} s^j x^{n+2-2j} \\
&\quad + q^{n+2} \sum_{j=0}^k q^{j^2} \left\{ \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + q^{n+1-2j+1} (1+q^j) \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right\} \frac{(-q; q)_{n+1-j}}{(-q; q)_j} s^j x^{n+2-2j}
\end{aligned}$$

We get the following result from Eq. (1.4) and Eq. (1.6)

$$\begin{aligned}
\mathcal{U}_{n+2}^k + q^{2n+3} s \mathcal{U}_n^{k-1} &= \sum_{j=0}^k q^{j^2} \frac{[n+2]_q}{[n+2-j]_q} \begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q \frac{(-q; q)_{n+1-j}}{(-q; q)_j} s^j x^{n+2-2j} \\
&\quad + q^{n+2} \sum_{j=0}^k q^{j^2} \frac{[n+2]_q}{[n+2-j]_q} \begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q \frac{(-q; q)_{n+1-j}}{(-q; q)_j} s^j x^{n+2-2j} \\
&= \mathcal{T}_{n+2}^k + q^{n+2} \mathcal{T}_{n+2}^k.
\end{aligned}$$

□

**Lemma 1.** We have

$$\frac{d \mathcal{U}_n}{dx} = nx^{-1} \mathcal{U}_n - 2x^{-1} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j} \quad (3.10)$$

and

$$\frac{d\mathcal{T}_n}{dx} = nx^{-1}\mathcal{T}_n - 2x^{-1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} jq^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j-1}}{(-q; q)_j} s^j x^{n-2j}. \quad (3.11)$$

*Proof.* By using Eq. (2.3), we have

$$\begin{aligned} \frac{d\mathcal{U}_n}{dx} &= \frac{d}{dx} \left\{ \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j} \right\} \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} (n-2j) \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j-1} \\ &= nx^{-1} \mathcal{U}_n - 2 \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} jq^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j-1}. \end{aligned}$$

Similarly, from Eq. (2.4), we get Eq. (3.11).  $\square$

Using Lemma 1, we can prove the following theorem.

**Theorem 8.** *We have*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_n^k = \left( \left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{U}_n + \frac{x}{2} \frac{d\mathcal{U}_n}{dx}. \quad (3.12)$$

*Proof.* From Eq. (3.1), we have

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_n^k &= \mathcal{U}_n^0 + \mathcal{U}_n^1 + \dots + \mathcal{U}_n^{\lfloor \frac{n}{2} \rfloor} \\ &= \left( q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q; q)_n}{(-q; q)_0} x^n \right) + \left( q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q; q)_n}{(-q; q)_0} x^n + q \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q; q)_{n-1}}{(-q; q)_1} s x^{n-2} \right) + \dots \\ &\quad + \left( q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q; q)_n}{(-q; q)_0} x^n + q \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q; q)_{n-1}}{(-q; q)_1} s x^{n-2} + \dots + q^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n - \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_q \frac{(-q; q)_{n - \lfloor \frac{n}{2} \rfloor}}{(-q; q)_{\lfloor \frac{n}{2} \rfloor}} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \right) \\ &= \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \left( q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q; q)_n}{(-q; q)_0} x^n \right) + \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 - 1 \right) \left( q \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q; q)_{n-1}}{(-q; q)_1} s x^{n-2} \right) + \dots \\ &\quad + \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 - \left\lfloor \frac{n}{2} \right\rfloor \right) \left( q^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n - \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_q \frac{(-q; q)_{n - \lfloor \frac{n}{2} \rfloor}}{(-q; q)_{\lfloor \frac{n}{2} \rfloor}} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \right) \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 - j \right) q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j} \\ &= \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \mathcal{U}_n - \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} jq^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j}}{(-q; q)_j} s^j x^{n-2j}. \end{aligned}$$

Then by using Lemma 1, we get

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_n^k = \left( \left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{U}_n + \frac{x}{2} \frac{d\mathcal{U}_n}{dx}.$$

$\square$



**Theorem 9.** *We have*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{T}_n^k = \left( \left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{T}_n + \frac{x}{2} \frac{d\mathcal{T}_n}{dx}. \quad (3.13)$$

*Proof.* We have from Eq. (3.2)

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{T}_n^k &= \mathcal{T}_n^0 + \mathcal{T}_n^1 + \dots + \mathcal{T}_n^{\lfloor \frac{n}{2} \rfloor} \\ &= (q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q; q)_{n-1}}{(-q; q)_0} x^n) + (q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q; q)_{n-1}}{(-q; q)_0} x^n + q \frac{[n]_q}{[n-1]_q} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q; q)_{n-2}}{(-q; q)_1} s x^{n-2}) + \dots \\ &\quad + \left( q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q; q)_{n-1}}{(-q; q)_0} x^n + q \frac{[n]_q}{[n-1]_q} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q; q)_{n-2}}{(-q; q)_1} s x^{n-2} + \dots \right. \\ &\quad \left. + q^{\lfloor \frac{n}{2} \rfloor^2} \frac{[n]_q}{[n - \lfloor \frac{n}{2} \rfloor]_q} \begin{bmatrix} n - \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_q \frac{(-q; q)_{n - \lfloor \frac{n}{2} \rfloor - 1}}{(-q; q)_{\lfloor \frac{n}{2} \rfloor}} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \right) \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 - j \right) q^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j-1}}{(-q; q)_j} s^j x^{n-2j} \\ &= \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \mathcal{T}_n - \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} j q^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q; q)_{n-j-1}}{(-q; q)_j} s^j x^{n-2j}. \end{aligned}$$

Lemma 1 implies that

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{T}_n^k = \left( \left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{T}_n + \frac{x}{2} \frac{d\mathcal{T}_n}{dx}.$$

□

## 4 Tables and Graphs of Incomplete $q$ -Chebyshev polynomials

In this section, we display the graphs of the  $q$ -Chebyshev polynomials and incomplete  $q$ -Chebyshev polynomials. Also we give the tables of some special cases of incomplete  $q$ -Chebyshev polynomials and numbers.

Table 3: Some special cases of the incomplete  $q$ -Chebyshev polynomials of the second kind

$x$	$s$	$q$	$\mathcal{U}_n^k(x, s, q)$	Incomplete $q$ -Chebyshev polynomials of the second kind
$x$	$-1$	$1$	$U_n^k(x)$	Incomplete Chebyshev polynomials of the second kind
$\frac{x}{2}$	$y$	$1$	$F_{n+1}^{(k)}(x, y)$	Incomplete bivariate Fibonacci polynomials
$\frac{x}{2}$	$1$	$1$	$F_{n+1}^{(k)}(x)$	Incomplete Fibonacci polynomials
$\frac{1}{2}$	$1$	$1$	$F_{n+1}^{(k)}$	Incomplete Fibonacci numbers
$x$	$1$	$1$	$P_{n+1}^{(k)}(x)$	Incomplete Pell polynomials
$1$	$1$	$1$	$P_{n+1}^{(k)}$	Incomplete Pell numbers
$\frac{1}{2}$	$2y$	$1$	$J_{n+1}^{(k)}(x)$	Incomplete Jacobsthal polynomials
$\frac{1}{2}$	$2$	$1$	$J_{n+1}^{(k)}$	Incomplete Jacobsthal numbers

The numerical results for the incomplete Chebyshev numbers, first and second kind, incomplete Fibonacci, incomplete Pell, and incomplete Jacobsthal numbers are displayed in Table 4-5. The numerical results for the incomplete Lucas, incomplete Pell-Lucas, and incomplete Jacobsthal-Lucas numbers are displayed in Table 6.

Table 4: Some special cases of the incomplete  $q$ -Chebyshev polynomials of the first kind

$x$	$s$	$q$	$T_n^k(x, s, q)$	Incomplete $q$ -Chebyshev polynomials of the first kind
$x$	$-1$	$1$	$T_n^k(x)$	Incomplete Chebyshev polynomials of the first kind
$\frac{x}{2}$	$y$	$1$	$\frac{1}{2}L_n^{(k)}(x, y)$	Incomplete Bivariate Lucas polynomials
$\frac{x}{2}$	$1$	$1$	$\frac{1}{2}L_n^{(k)}(x)$	Incomplete Lucas polynomials
$\frac{1}{2}$	$1$	$1$	$\frac{1}{2}L_n^{(k)}$	Incomplete Lucas numbers
$x$	$1$	$1$	$\frac{1}{2}Q_n^{(k)}(x)$	Incomplete Pell-Lucas polynomials
$1$	$1$	$1$	$\frac{1}{2}Q_n^{(k)}$	Incomplete Pell-Lucas numbers
$\frac{1}{2}$	$2y$	$1$	$\frac{1}{2}j_n^{(k)}(y)$	Incomplete Jacobsthal-Lucas polynomials
$\frac{1}{2}$	$2$	$1$	$\frac{1}{2}j_n^{(k)}$	Incomplete Jacobsthal-Lucas numbers

Table 5: Incomplete Chebyshev numbers of the first and second kind

$n/k$	$\mathcal{T}_n^k(1, -1, 1)$					$\mathcal{U}_n^k(1, -1, 1)$				
	0	1	2	3	4	0	1	2	3	4
1	1					2				
2	2	1				4	3			
3	4	1				8	4			
4	8	0	1			16	4	5		
5	16	-4	1			32	0	6		
6	32	-16	2	1		64	-16	8	7	
7	64	-48	8	1		128	-64	16	8	
8	128	-128	32	0	1	256	-192	48	8	9
9	256	-320	112	-8	1	512	-512	160	0	10

Table 6: Incomplete Fibonacci numbers, incomplete Pell numbers and incomplete Jacobsthal numbers

$n/k$	$\mathcal{U}_n^k(\frac{1}{2}, 1, 1) = F_{n+1}^{(k)}$					$\mathcal{U}_n^k(1, 1, 1) = P_{n+1}^{(k)}$					$\mathcal{U}_n^k(\frac{1}{2}, 2, 1) = J_{n+1}^{(k)}$				
	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	1					2					1				
2	1	2				4	5				1	3			
3	1	3				8	12				1	5			
4	1	4	5			16	28	29			1	7	11		
5	1	5	8			32	64	70			1	9	21		
6	1	6	12	13		64	144	168	169		1	114	35	43	
7	1	7	17	21		128	320	400	408		1	13	53	85	
8	1	8	23	33	34	256	704	944	984	985	1	15	75	155	171
9	1	9	30	50	55	512	1536	2208	2368	2378	1	17	101	261	341

In Figures 1, 2 the graphs of the  $q$ -Chebyshev polynomials of first and second kind for  $s = -1$ ,  $q = -0.5, 0.5, 0.9, 0.9999$ ,  $n = 0, 1, 2, 3, 4, 5$  and  $-1 \leq x \leq 1$  are shown.

Table 7: Incomplete Lucas numbers, incomplete Pell-Lucas numbers and incomplete Jacobsthal-Lucas numbers

$n/k$	$2\mathcal{T}_n^k(\frac{1}{2}, 1, 1) = L_n^{(k)}$					$2\mathcal{T}_n^k(1, 1, 1) = Q_n^{(k)}$					$2\mathcal{T}_n^k(\frac{1}{2}, 2, 1) = j_n^{(k)}$				
	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	1					2					1				
2	1	3				4	6				1	5			
3	1	4					8	14			1	7			
4	1	5	7			16	32	34			1	9	17		
5	1	6	11			32	72	82			1	11	31		
6	1	7	16	18		64	160	196	198		1	13	49	65	
7	1	8	22	29		128	352	464	478		1	15	71	127	
8	1	9	29	45	47	256	768	1088	1152	1154	1	17	97	225	257
9	1	10	37	67	76	512	1664	2528	2768	2786	1	19	127	367	511

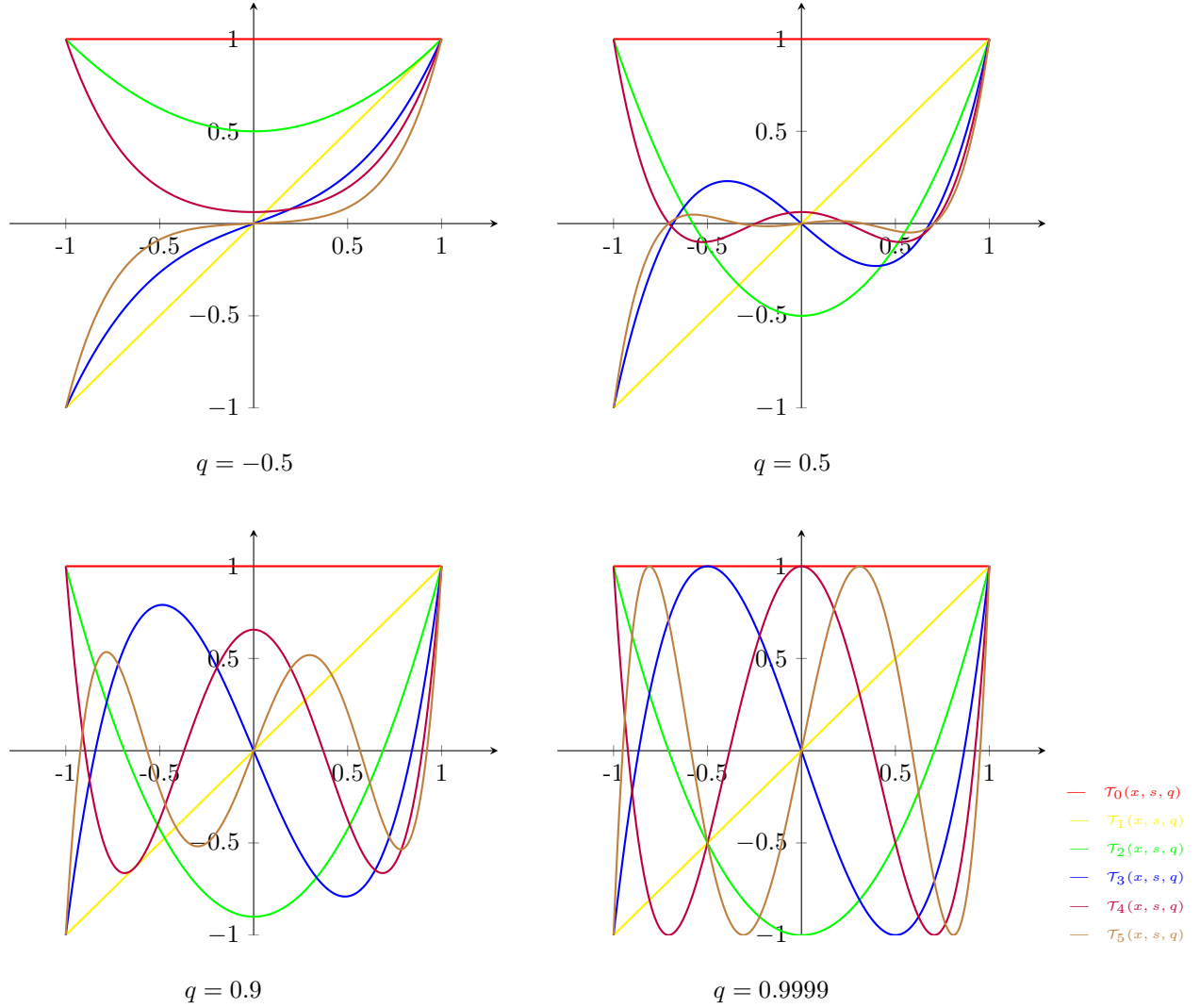


Figure 1: Graphs of  $\mathcal{T}_n(x, s, q)$  for  $s = -1, q = -0.5, 0.5, 0.9, 0.9999$ ,  $n = 0, 1, 2, 3, 4, 5$

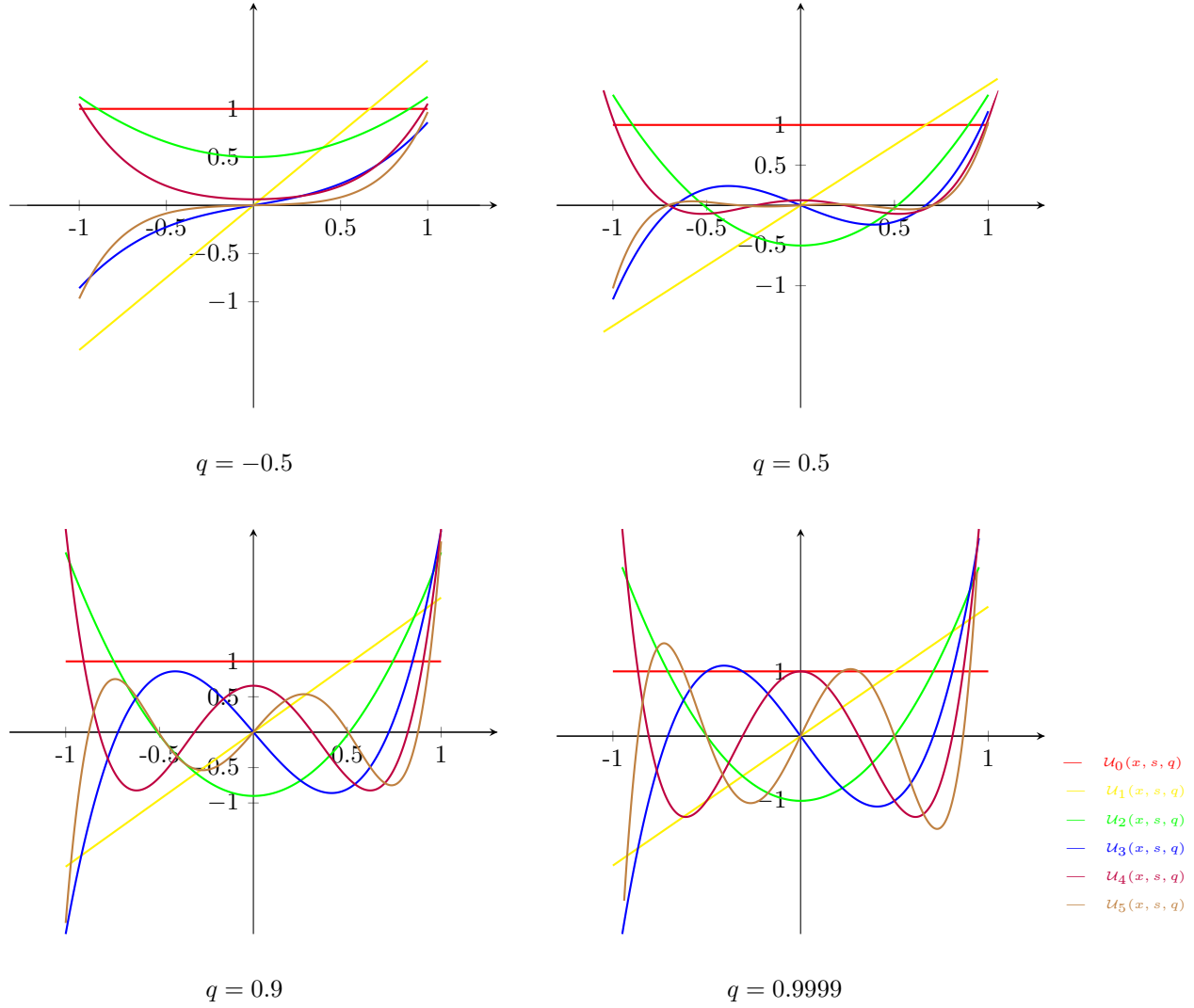


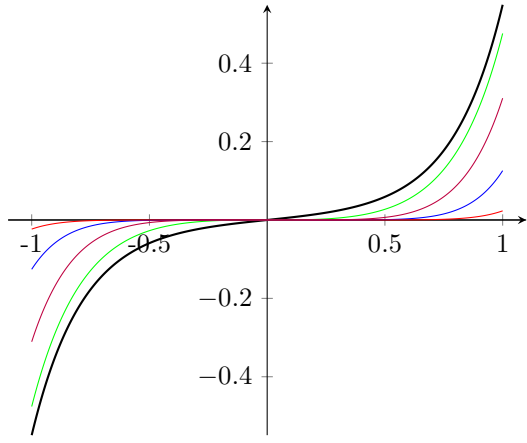
Figure 2: Graphs of  $\mathcal{U}_n(x, s, q)$  for  $s = -1$ ,  $q = -0.5, 0.5, 0.9, 0.9999$ ,  $n = 0, 1, 2, 3, 4, 5$

In Figure 3 the graphs of the incomplete  $q$ -Chebyshev polynomials of second kind  $\mathcal{U}_9^k(x, s, q)$  for  $s = -1$ ,  $q = -0.9, -0.5, 0.5, 0.9$ ,  $k = 0, 1, 2, 3, 4$  are shown.

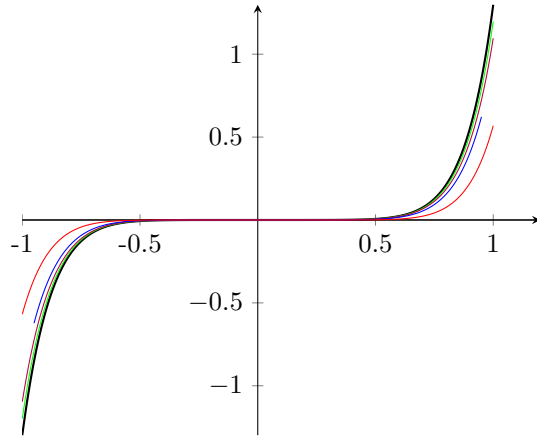
In Figure 4 the graphs of the incomplete Lucas polynomials  $\mathcal{T}_5^k(\frac{x}{2}, s, q)$  for  $s = 1$ ,  $q = -0.9, -0.5, 0.5, 0.9$ ,  $k = 0, 1, 2$  are shown.

In Figure 5 the graphs of the incomplete Jacobsthal polynomials  $\mathcal{U}_8^k(\frac{x}{2}, s, q)$  for  $s = 2$ ,  $q = -0.9, -0.5, 0.5, 0.9$ ,  $k = 0, 1, 2, 3, 4$  are shown.

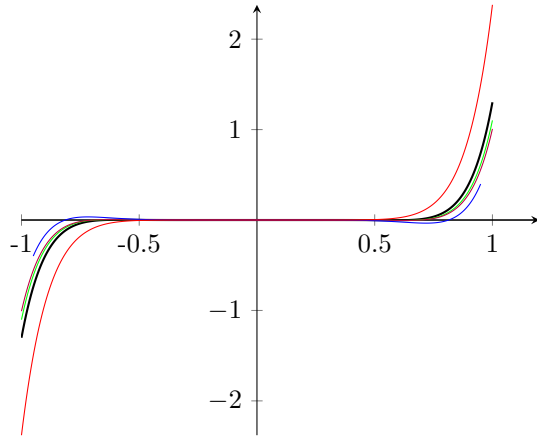
In Figure 6, the graphs of incomplete Fibonacci numbers  $\mathcal{U}_n^k(\frac{x}{2}, 1, 1)$  and incomplete Lucas numbers  $\mathcal{T}_n^k(\frac{x}{2}, 1, 1)$  for  $1 \leq n \leq 9$   $0 \leq k \leq k$  are shown.



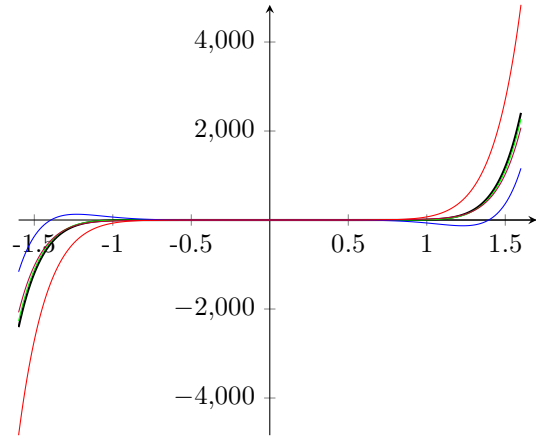
$q = -0.9$



$q = -0.5$



$q = 0.5$



$q = 0.9$

$\text{--- } \mathcal{U}_9^0(x, s, q)$   
 $\text{--- } \mathcal{U}_9^1(x, s, q)$   
 $\text{--- } \mathcal{U}_9^2(x, s, q)$   
 $\text{--- } \mathcal{U}_9^3(x, s, q)$   
 $\text{--- } \mathcal{U}_9^4(x, s, q)$

Figure 3: Graphs of  $\mathcal{U}_9^k(x, s, q)$  for  $s = -1$ ,  $q = -0.9, -0.5, 0.5, 0.9$ ,  $k = 0, 1, 2, 3, 4$

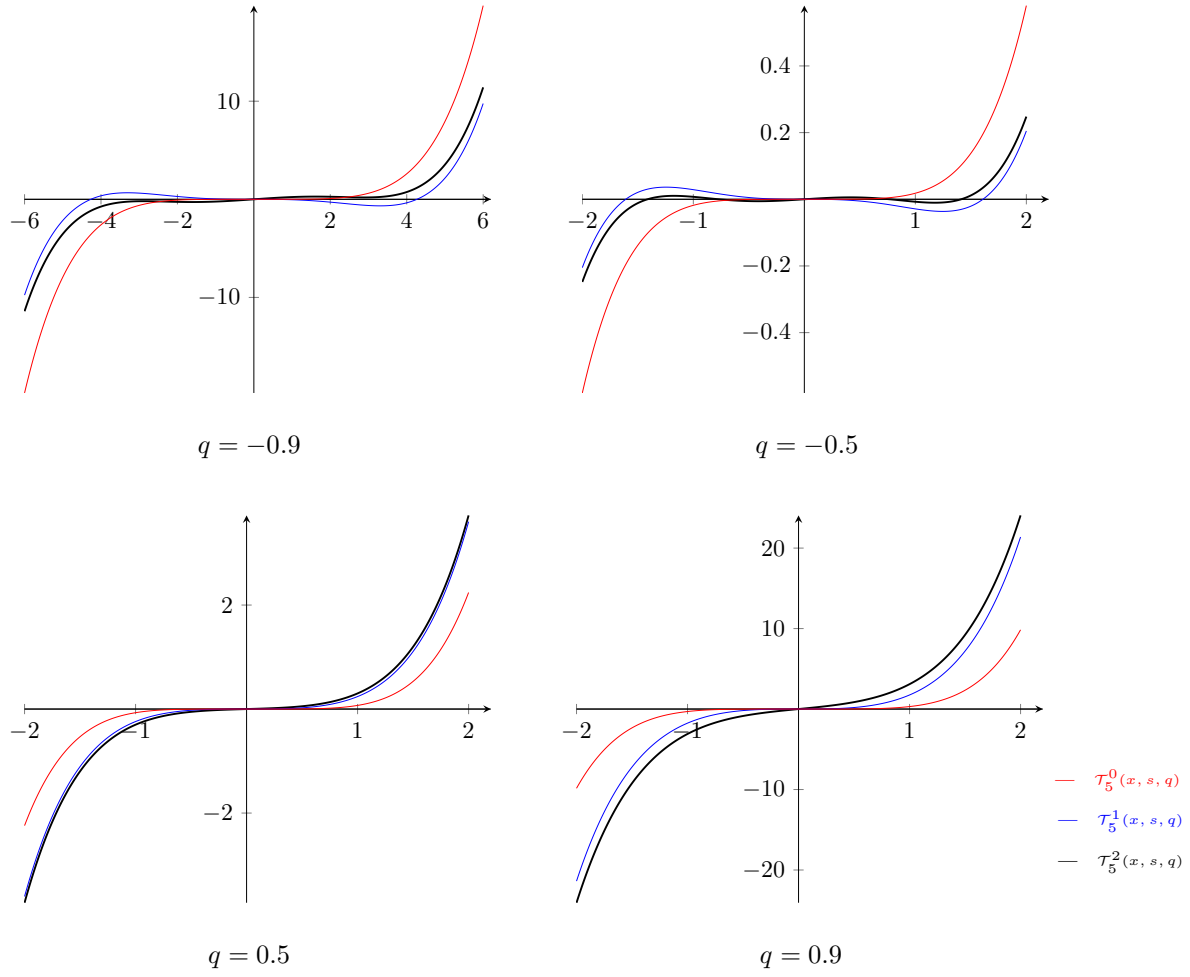


Figure 4: Graphs of  $\mathcal{T}_5^k(\frac{x}{2}, s, q)$  for  $s = 1$ ,  $q = -0.9, -0.5, 0.5, 0.9$ ,  $k = 0, 1, 2$

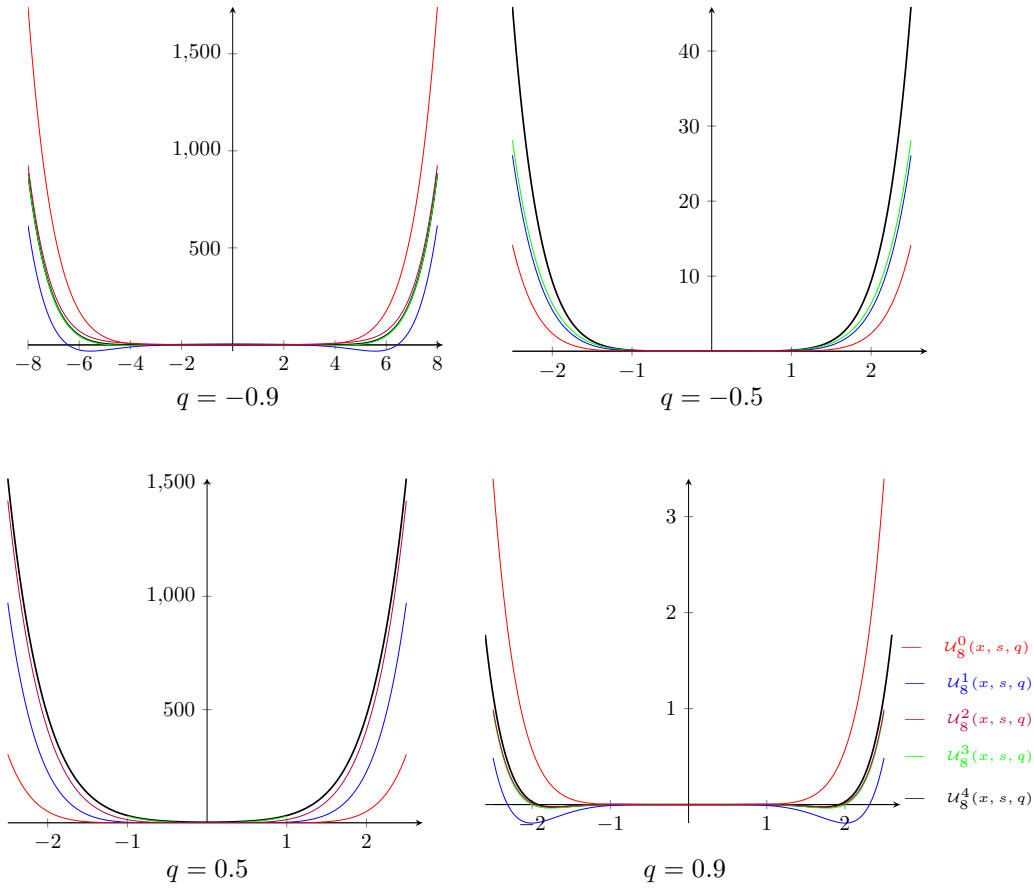


Figure 5: Graphs of  $\mathcal{U}_8^k(\frac{x}{2}, s, q)$   $s = 2, q = -0.9, -0.5, 0.5, 0.9, k = 0, 1, 2, 3, 4$

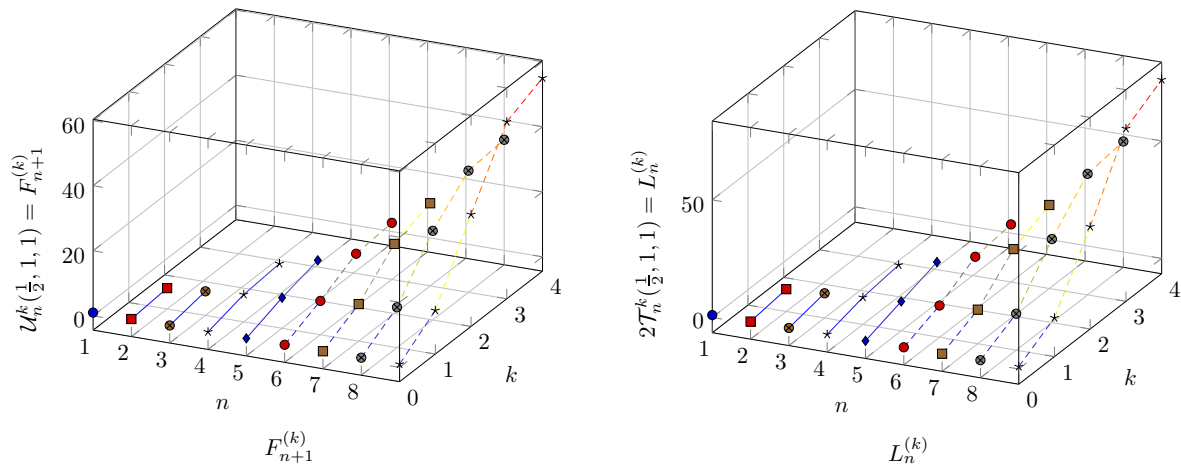


Figure 6: Graphs of incomplete Fibonacci ve Lucas numbers for  $1 \leq n \leq 9, 0 \leq k \leq 4$ .

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